

Traveling Wave Solutions of Spatially Periodic Nonlocal Monostable Equations*

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Abstract. This paper deals with front propagation dynamics of monostable equations with nonlocal dispersal in spatially periodic habitats. In the authors' earlier works, it is shown that a general spatially periodic monostable equation with nonlocal dispersal has a unique spatially periodic positive stationary solution and has a spreading speed in every direction. In this paper, we show that a spatially periodic nonlocal monostable equation with certain spatial homogeneity or small nonlocal dispersal distance has a unique stable periodic traveling wave solutions connecting its unique spatially periodic positive stationary solution and the trivial solution in every direction for all speeds greater than the spreading speed in that direction.

Key words. Monostable equation; nonlocal dispersal; random dispersal; spreading speed; traveling wave solution; principal eigenvalue; principal eigenfunction.

Mathematics subject classification. 35K55, 45C05, 45G10, 45M20, 47G20, 92D25.

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1 Introduction

The current paper is concerned with traveling wave solutions of spatially periodic nonlocal monostable equations.

Monostable equations arise in modeling population dynamics of species in biology and ecology. Classically, one assumes that the internal interaction of species is random and local (i.e. species moves randomly between the adjacent spatial locations), which leads to the following reaction-diffusion equation,

$$\frac{\partial u}{\partial t} = \Delta u + u f(x, u), \quad x \in \mathbb{R}^N, \quad (1.1)$$

where $u(t, x)$ represents the population density of species at time t and spatial location x and $f(x, u)$ satisfies certain monostability assumptions. Roughly, the monostability assumptions mean that $f(x, u) < 0$ for $u \gg 1$, $f_u(x, u) < 0$ for $u \geq 0$, and the trivial solution $u = 0$ is unstable.

In reality, the movements and interactions of many species in biology and ecology can occur between non-adjacent spatial locations. Taking the nonlocal internal interaction of species into the account leads to the following nonlocal dispersal evolution equation,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y - x) u(t, y) dy - u(t, x) + u(t, x) f(x, u(t, x)), \quad x \in \mathbb{R}^N, \quad (1.2)$$

where $k(\cdot)$ is a C^1 convolution kernel supported on a ball centered at the origin (that is, there is a $\delta_0 > 0$ such that $k(z) > 0$ if $\|z\| < \delta_0$, $k(z) = 0$ if $\|z\| \geq \delta_0$, where $\|\cdot\|$ denotes the norm in \mathbb{R}^N and δ_0 represents the nonlocal dispersal distance), $\int_{\mathbb{R}^N} k(z) dz = 1$, and $f(x, u)$ satisfies certain monostable assumptions.

Throughout this paper, we assume that $f(x, u)$ is periodic in x with period vector $\mathbf{p} = (p_1, p_2, \dots, p_N)$ (that is, $f(\cdot + p_i \mathbf{e}_i, \cdot) = f(\cdot, \cdot)$, $\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{iN})$, $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, $i, j = 1, 2, \dots, N$). To state the monostability assumptions on f , let

$$X_p = \{u \in C(\mathbb{R}^N, \mathbb{R}) | u(\cdot + p_i \mathbf{e}_i) = u(\cdot), \quad i = 1, \dots, N\} \quad (1.3)$$

with norm $\|u\|_{X_p} = \sup_{x \in \mathbb{R}^N} |u(x)|$, and

$$X_p^+ = \{u \in X_p | u(x) \geq 0 \quad \forall x \in \mathbb{R}^N\}. \quad (1.4)$$

Let I be the identity map on X_p , and \mathcal{K} , $a_0(\cdot)I : X_p \rightarrow X_p$ be defined by

$$(\mathcal{K}u)(x) = \int_{\mathbb{R}^N} k(y - x) u(y) dy, \quad (1.5)$$

$$(a_0(\cdot)Iu)(x) = a_0(x)u(x), \quad (1.6)$$

where $a_0(x) = f(x, 0)$. We assume the following monostability assumptions for (1.1):

(A1) $f \in C^1(\mathbb{R}^N \times [0, \infty), \mathbb{R})$, $\sup_{x \in \mathbb{R}^N, u \geq 0} \frac{\partial f(x, u)}{\partial u} < 0$ and $f(x, u) < 0$ for $x \in \mathbb{R}^N$ and $u \gg 1$.

(A2) $u \equiv 0$ is linearly unstable in X_p , that is, the principal eigenvalue of

$$\begin{cases} \Delta u + a_0(x)u = \lambda u, & x \in \mathbb{R}^N \\ u(x + p_i \mathbf{e}_i) = u(x), & i = 1, 2, \dots, N, \ x \in \mathbb{R}^N \end{cases}$$

is positive.

The following are monostability assumptions for (1.2):

(H1) $f \in C^1(\mathbb{R}^N \times [0, \infty), \mathbb{R})$, $\sup_{x \in \mathbb{R}^N, u \geq 0} \frac{\partial f(x, u)}{\partial u} < 0$ and $f(x, u) < 0$ for $x \in \mathbb{R}^N$ and $u \gg 1$.

(H2) $u \equiv 0$ is linearly unstable in X_p , that is, $\lambda_0 := \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(\mathcal{K} - I + a_0(\cdot)I)\}$ is positive, where $\sigma(\mathcal{K} - I + a_0(\cdot)I)$ is the spectrum of the operator $\mathcal{K} - I + a_0(\cdot)I$ on X_p .

It is well known that (A1) and (A2) imply that (1.1) has exactly two equilibrium solutions in X_p^+ , $u = 0$ and $u = u^+$, and $u = 0$ is linearly unstable and $u = u^+$ is asymptotically stable in X_p , which reflects the monostable feature of the assumptions (A1) and (A2).

Observe that (A1) and (H1) are exactly the same (it is for convenience to state them separately). (H2) is the analogue of (A2). It should be pointed out that λ_0 in (H2) may not be an eigenvalue of $\mathcal{K} - I + a_0(\cdot)I$ (see an example in [54]) and therefore there is some essential difference between random dispersal and nonlocal dispersal operators. Nevertheless, it is proved in [55] that (H1) and (H2) also imply that (1.2) has exactly two equilibrium solutions in X_p^+ , $u = 0$ and $u = u^+$, and $u = 0$ is linearly unstable and $u = u^+$ is asymptotically stable in X_p (see Proposition 2.1 or [55, Theorem C]), which reflects the monostable feature of the assumptions (H1) and (H2).

Among the most important dynamical issues about monostable equations (1.1) and (1.2) are spatial spread and front propagation dynamics.

The spatial spread and front propagation dynamics of (1.1) has been extensively studied since the pioneering works by Fisher [17] and Kolmogorov, Petrowsky, Piscunov [35] on the following special case of (1.1)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u), \quad x \in \mathbb{R}, \quad (1.7)$$

which models the evolutionary take-over of a habitat by a fitter genotype. See, for example, [1], [3], [4], [5], [16], [18], [23], [27], [28], [33], [37], [38], [39], [40], [42], [43], [44], [47], [48], [49], [51], [52], [56], [57], [58], and references therein, for the study of the spatial spread and front propagation dynamics of (1.1). It is proved that, if (A1) and (A2) hold, then for every $\xi \in S^{N-1} := \{\xi \in \mathbb{R}^N \mid \|\xi\| = 1\}$, there is a $c^*(\xi) \in \mathbb{R}$ such that for every $c \geq c^*(\xi)$, there is a traveling wave solution connecting u^+ and $u^- \equiv 0$ and propagating in the direction of ξ with speed c , and there is no such traveling wave solution of slower speed in the direction of ξ . Moreover, the minimal wave speed $c^*(\xi)$ is of some important spreading properties (hence is also called the spreading speed in the direction of ξ) and has the following variational characterization.

Let $\lambda(\xi, \mu)$ be the eigenvalue of

$$\begin{cases} \Delta u - 2\mu \sum_{i=1}^N \xi_i \frac{\partial u}{\partial x_i} + (a_0(x) + \mu^2)u = \lambda u, & x \in \mathbb{R}^N \\ u(x + p_i \mathbf{e}_i) = u(x), & i = 1, 2, \dots, N \quad x \in \mathbb{R}^N \end{cases} \quad (1.8)$$

with largest real part, where $a_0(x) = f(x, 0)$ (it is well known that $\lambda(\xi, \mu)$ is real and algebraically simple. $\lambda(\xi, \mu)$ is called the *principal eigenvalue* of (1.8) in literature). Then

$$c^*(\xi) = \inf_{\mu > 0} \frac{\lambda(\xi, \mu)}{\mu}. \quad (1.9)$$

(See [3], [4], [5], [37], [42], [43], [44], [58] and references therein for the above mentioned properties).

Recently, various dynamical problems related to the spatial spread and front propagation dynamics of nonlocal dispersal equations of the form (1.2) have also been studied by many authors. See, for example, [2], [6], [10], [12], [14], [19], [20], [25], [26], [30], [31], [32], [34], [53], for the study of spectral theory for nonlocal dispersal operators and the existence, uniqueness, and stability of nontrivial positive stationary solutions. See, for example, [11], [13], [15], [36], [41], [45], [57], [58], for the study of entire solutions and the existence of spreading speeds and traveling wave solutions connecting the trivial solution $u = 0$ and a nontrivial positive stationary solution for some special cases of (1.2). In particular, if $f(x, u)$ is independent of x , then it is proved that (1.2) has a spreading speed $c^*(\xi)$ in every direction of $\xi \in S^{N-1}$ ($c^*(\xi)$ is indeed independent of $\xi \in S^{N-1}$ in this case) and for every $c \geq c^*(\xi)$, (1.2) has a traveling wave solution connecting u^+ and 0 and propagating in the direction of ξ with propagating speed c (see [11]). In the very recent papers [54], [55], the authors of the current paper explored the spatial spread dynamics of general spatially periodic monostable equations and proved that assume (H1) and (H2), (1.2) has a spreading speed $c^*(\xi)$ in every direction of $\xi \in S^{N-1}$, which extends the existing results on spreading speed of (1.1) to (1.2).

However, there is little understanding about traveling wave solutions of spatially periodic monostable equations with nonlocal dispersal. The objective of the current paper is to investigate the existence, uniqueness, and stability of traveling wave solutions of (1.2). We show that if the periodic habitat of (1.2) is of certain homogeneity or the nonlocal dispersal distance is small, then (1.2) has a unique stable traveling wave solution which connects u^+ and 0 and propagates in a given direction $\xi \in S^{N-1}$ for all speeds greater than the spreading speed in the direction of ξ . The main tools employed in the proofs of the existence, uniqueness, and stability of traveling wave solutions of (1.2) include sub- and super-solutions and the principal eigenvalue theory for nonlocal dispersal operators which has recently been established in [54].

It should be pointed out that the spatial spread and front propagation dynamics of spatially discrete monostable equations has also been widely studied. The reader is referred to [7], [8], [9], [21], [22], [29], [50], [59], [60].

The rest of this paper is organized as follows. In section 2, we introduce some standing notations and the definition of spatially periodic traveling wave solutions and state the main

results of the paper. In section 3, we present the comparison principle for solutions of (1.2) and some related linear equations with nonlocal dispersal and construct some sub- and super-solutions of (1.2) to be used in the proofs of the main results. The existence of traveling wave solutions is investigated in section 4. Section 5 concerns the uniqueness and continuity of traveling wave solutions. In section 6, we show the stability of traveling wave solutions.

2 Notations, Definitions, and Main Results

In this section, we introduce some standing notations and the definition of spatially periodic traveling wave solutions, and state the main results of the paper.

First of all, let X_p and X_p^+ be as in (1.3) and (1.4), respectively. Let

$$X = \{u \in C(\mathbb{R}^N, \mathbb{R}) \mid u \text{ is uniformly continuous on } \mathbb{R}^N \text{ and } \sup_{x \in \mathbb{R}^N} |u(x)| < \infty\} \quad (2.1)$$

with norm $\|u\|_X = \sup_{x \in \mathbb{R}^N} |u(x)|$, and

$$X^+ = \{u \in X \mid u(x) \geq 0 \quad \forall x \in \mathbb{R}^N\}. \quad (2.2)$$

Let

$$\tilde{X} = \{u : \mathbb{R}^N \rightarrow \mathbb{R} \mid u \text{ is Lebesgue measurable and bounded}\} \quad (2.3)$$

endowed with the norm $\|u\|_{\tilde{X}} = \sup_{x \in \mathbb{R}^N} |u(x)|$ and

$$\tilde{X}^+ = \{u \in \tilde{X} \mid u(x) \geq 0 \quad \forall x \in \mathbb{R}^N\}. \quad (2.4)$$

Observe that $X_p \subset X \subset \tilde{X}$.

Consider the shifted equations of (1.2),

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + u(t,x)f(x+z, u(t,x)), \quad x \in \mathbb{R}^N \quad (2.5)$$

where $z \in \mathbb{R}^N$. By general semigroup theory (see [24] and [46]), for any $u_0 \in \tilde{X}$ and $z \in \mathbb{R}$, (2.5) has a unique (local) solution $u(t, \cdot) \in \tilde{X}$ with $u(0, x) = u_0(x)$. Let $u(t, x; u_0, z)$ be the solution of (2.5) with $u(0, x; u_0, z) = u_0(x)$. Note that if $u_0 \in X_p$ (resp. X), then $u(t, \cdot; u_0, z) \in X_p$ (resp. X). If $u_0 \in \tilde{X}^+$, then $u(t, x; u_0)$ exists for all $t \geq 0$ (see Proposition 3.1).

A measurable function $u : \mathbb{R} \times \mathbb{R}^N$ is called an *entire solution* of (1.2) if $u(t, x)$ is differentiable in $t \in \mathbb{R}$ and satisfies (1.2) for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

Proposition 2.1. *Assume (H1)-(H2). Then (1.2) has a unique stationary solution $u^+(\cdot) \in X_p^+ \setminus \{0\}$. Moreover, $u = u^+(\cdot)$ is asymptotically stable with respect to perturbations in $X_p^+ \setminus \{0\}$ and for any $\xi \in S^{N-1}$, any $u_0 \in \tilde{X}^+$, $u_0(x) \geq \delta$ for all $x \in \mathbb{R}^N$ with $x \cdot \xi \leq m$ for some $m \in \mathbb{R}$ and $\delta > 0$, and any $\epsilon > 0$, there are $T > 0$ and $R \in \mathbb{R}$ such that*

$$\sup_{x, z \in \mathbb{R}^N, x \cdot \xi \leq r} |u(T, x; u_0, z) - u^+(x+z)| < \epsilon \quad \forall r \leq R.$$

Proof. It follows from the arguments in [55, Theorem C] and [54, Proposition 2.3]. \square

For given function $g : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$, $\xi \in S^{N-1}$, and $c, \alpha \in \mathbb{R}$, we define the following limit:

$$\lim_{x \cdot \xi - ct \rightarrow \infty (-\infty)} g(t, x, z) = \alpha \text{ uniformly in } z \in \mathbb{R}^N$$

if and only if

$$\lim_{r \rightarrow \infty (-\infty)} \sup_{t \in \mathbb{R}, x, z \in \mathbb{R}^N, x \cdot \xi - ct \geq r (\leq r)} |g(t, x, z) - \alpha| = 0.$$

Definition 2.1 (Traveling wave solution). (1) An entire solution $u(t, x)$ of (1.2) is called a traveling wave solution connecting $u^+(\cdot)$ and 0 and propagating in the direction of ξ with speed c if there is a bounded measurable function $\Phi : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^+$ such that $u(t, \cdot; \Phi(\cdot, z), z)$ exists for all $t \in \mathbb{R}$,

$$u(t, x) = u(t, x; \Phi(\cdot, 0), 0) = \Phi(x - ct\xi, ct\xi) \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^N, \quad (2.6)$$

$$u(t, x; \Phi(\cdot, z), z) = \Phi(x - ct\xi, z + ct\xi) \quad \forall t \in \mathbb{R}, x, z \in \mathbb{R}^N, \quad (2.7)$$

$$\lim_{x \cdot \xi \rightarrow -\infty} (\Phi(x, z) - u^+(x + z)) = 0, \quad \lim_{x \cdot \xi \rightarrow \infty} \Phi(x, z) = 0 \quad \text{uniformly in } z \in \mathbb{R}^N, \quad (2.8)$$

$$\Phi(x, z - x) = \Phi(x', z - x') \quad \forall x, x' \in \mathbb{R}^N \text{ with } x \cdot \xi = x' \cdot \xi, \quad (2.9)$$

and

$$\Phi(x, z + p_i \mathbf{e}_i) = \Phi(x, z) \quad \forall x, z \in \mathbb{R}^N. \quad (2.10)$$

(2) A bounded measurable function $\Phi : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^+$ is said to generate a traveling wave solution of (1.2) in the direction of ξ with speed c if it satisfies (2.7)-(2.10).

Remark 2.1. Suppose that $u(t, x) = \Phi(x - ct\xi, ct\xi)$ is a traveling wave solution of (1.2) connecting $u^+(\cdot)$ and 0 and propagating in the direction of ξ with speed c . Then $u(t, x)$ can be written as

$$u(t, x) = \Psi(x \cdot \xi - ct, x) \quad (2.11)$$

for some $\Psi : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfying that $\Psi(\eta, z + p_i \mathbf{e}_i) = \Psi(\eta, z)$, $\lim_{\eta \rightarrow -\infty} \Psi(\eta, z) = u^+(z)$, and $\lim_{\eta \rightarrow \infty} \Psi(\eta, z) = 0$ uniformly in $z \in \mathbb{R}^N$. In fact, let $\Psi(\eta, z) = \Phi(x, z - x)$ for $x \in \mathbb{R}^N$ with $x \cdot \xi = \eta$. Observe that $\Psi(\eta, z)$ is well defined and has the above mentioned properties. In some literature, the form (2.11) is adopted for spatially periodic traveling wave solutions (see [37], [42], [58], and references therein).

Next, we recall some principal eigenvalue theory and spatial spreading theory established in [54] and [55].

Consider the following eigenvalue problem, which is a nonlocal counterpart of (1.8),

$$(\mathcal{K}_{\xi, \mu} - I + a(\cdot)I)v = \lambda v, \quad v \in X_p, \quad (2.12)$$

where $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, and $a(\cdot) \in X_p$. The operator $a(\cdot)I$ has the same meaning as in (1.6) with $a_0(\cdot)$ being replaced by $a(\cdot)$, and $\mathcal{K}_{\xi,\mu} : X_p \rightarrow X_p$ is defined by

$$(\mathcal{K}_{\xi,\mu}v)(x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)v(y)dy. \quad (2.13)$$

We point out the following relation between (1.2) and (2.12): if $u(t, x) = e^{-\mu(x\cdot\xi - \frac{\lambda}{\mu}t)}\phi(x)$ with $\phi \in X_p \setminus \{0\}$ is a solution of the linearization of (1.2) at $u = 0$,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t, y)dy - u(t, x) + a_0(x)u(t, x), \quad x \in \mathbb{R}^N, \quad (2.14)$$

where $a_0(x) = f(x, 0)$, then λ is an eigenvalue of (2.12) with $a(\cdot) = a_0(\cdot)$ or $\mathcal{K}_{\xi,\mu} - I + a_0(\cdot)I$ and $v = \phi(x)$ is a corresponding eigenfunction.

Let $\sigma(\mathcal{K}_{\xi,\mu} - I + a(\cdot)I)$ be the spectrum of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ on X_p . Let

$$\lambda_0(\xi, \mu, a) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathcal{K}_{\xi,\mu} - I + a(\cdot)I)\}.$$

We call $\lambda_0(\xi, \mu, a)$ the *principal spectrum point* of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$. Observe that if $\mu = 0$, (2.12) is independent of ξ and hence we put

$$\lambda_0(a) := \lambda_0(\xi, 0, a) \quad \forall \xi \in S^{N-1}. \quad (2.15)$$

$\lambda_0(\xi, \mu, a)$ is called the *principal eigenvalue* of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ or $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ is said to have a principal eigenvalue if $\lambda_0(\xi, \mu, a)$ is an algebraically simple eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ with an eigenfunction $v \in X_p^+$, and for every $\lambda \in \sigma(\mathcal{K}_{\xi,\mu} - I + a(\cdot)I) \setminus \{\lambda_0(\xi, \mu, a)\}$, $\operatorname{Re}\lambda < \lambda_0(\xi, \mu, a)$.

Observe that $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ may not have a principal eigenvalue (see an example in [54]), which reveals some essential difference between random dispersal operators and nonlocal dispersal operators. The following proposition on the existence of principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ is proved in [54] (see also [55]).

Proposition 2.2. (1) If $k(x) = \frac{1}{\delta^N} \tilde{k}(\frac{x}{\delta})$ for all $x \in \mathbb{R}^N$, where $\tilde{k}(\cdot)$ satisfies that $\tilde{k}(z) > 0$ for $\|z\| < 1$, $\tilde{k}(z) = 0$ for $\|z\| \geq 1$, and $\int_{\mathbb{R}^N} \tilde{k}(z)dz = 1$, then $\lambda_0(\xi, \mu, a)$ is the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ for all $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$ and $0 < \delta \ll 1$.

(2) If $a(x)$ satisfies that $\max_{x \in \mathbb{R}^N} a(x) - \min_{x \in \mathbb{R}^N} a(x) < 1$, then $\lambda_0(\xi, \mu, a)$ is the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ for all $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$.

(3) If $a(\cdot)$ is C^N and the partial derivatives of $a(x)$ up to order $N-1$ at some x_0 are zero, where x_0 is such that $a(x_0) = \max_{x \in \mathbb{R}^N} a(x)$, then the conclusion in (2) holds.

Proposition 2.2 shows such an important fact: nonlocal dispersal operator possesses a similar principal eigenvalue theory to random dispersal operator for following cases: the nonlocal dispersal is nearly local; the periodic habitat is *nearly globally homogeneous* (in the sense that

the condition in Proposition 2.2(2) is satisfied) or it is *nearly homogeneous in a region where it is most conducive to population growth* in the zero-limit population (in the sense that the condition in Proposition 2.2(3) is satisfied). Note that if $a_0(\cdot)$ is C^1 and $1 \leq N \leq 2$, the condition in Proposition 2.2(3) is always satisfied.

As it is mentioned above, a spatially periodic monostable equation with random dispersal has a spreading speed in every direction. This important feature has been well extended in [54] and [55] to spatially periodic monostable equations with nonlocal dispersal. For given function $h : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, we define

$$\begin{aligned} \liminf_{x \cdot \xi \rightarrow -\infty} h(t, x, z) &= \liminf_{r \rightarrow -\infty} \inf_{x \in \mathbb{R}^N, x \cdot \xi \leq r} h(t, x, z), \\ \limsup_{x \cdot \xi \rightarrow \infty} h(t, x, z) &= \limsup_{r \rightarrow \infty} \sup_{x \in \mathbb{R}^N, x \cdot \xi \geq r} h(t, x, z), \\ \liminf_{t \rightarrow \infty} \inf_{x \cdot \xi \leq ct} h(t, x, z) &= \liminf_{t \rightarrow \infty} \inf_{x \in \mathbb{R}^N, x \cdot \xi \leq ct} h(t, x, z), \end{aligned}$$

and

$$\limsup_{t \rightarrow \infty} \sup_{x \cdot \xi \geq ct} h(t, x, z) = \limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^N, x \cdot \xi \geq ct} h(t, x, z).$$

Roughly speaking, a number $c^*(\xi) \in \mathbb{R}$ is called the spreading speed of (1.2) in the direction of ξ if for every $u_0 \in X^+$ with $\liminf_{x \cdot \xi \rightarrow -\infty} u_0(x) > 0$ and $u_0(x) = 0$ for $x \cdot \xi \gg 1$,

$$\liminf_{t \rightarrow \infty} \inf_{x \cdot \xi \leq ct} (u(t, x; u_0) - u^+(x)) = 0 \quad \forall c < c^*(\xi)$$

and

$$\limsup_{t \rightarrow \infty} \sup_{x \cdot \xi \geq ct} u(t, x; u_0) = 0 \quad \forall c > c^*(\xi)$$

(see [55, Definition 1.2] for detail). The following proposition on the existence of spreading speeds is proved in [55] (see also [54]).

Proposition 2.3. *Assume (H1) and (H2). For any $\xi \in S^{N-1}$, (1.2) has a spreading speed $c^*(\xi)$ in the direction of ξ . Moreover, there is $\mu^*(\xi) > 0$ such that*

$$c^*(\xi) = \inf_{\tilde{\mu} > 0} \frac{\lambda_0(\xi, \tilde{\mu}, a_0)}{\tilde{\mu}} = \frac{\lambda_0(\xi, \mu^*(\xi), a_0)}{\mu^*(\xi)} < \frac{\lambda_0(\xi, \mu, a_0)}{\mu} \quad \forall \mu \in (0, \mu^*(\xi)).$$

For convenience, we introduce the following standing assumption:

(H3) *For every $\xi \in S^{N-1}$ and $\mu \geq 0$, $\lambda_0(\xi, \mu, a_0)$ is the principal eigenvalue of $\mathcal{K}_{\xi, \mu} - I + a_0(\cdot)I$, where $a_0(x) = f(x, 0)$.*

Biologically, one is only interested in nonnegative solutions of (1.2). Without loss of generality, we then also assume

(H4) $f(x, u) = f(x, 0)$ for $u \leq 0$.

We now state the main results of the paper. For given $\xi \in S^{N-1}$ and $c > c^*(\xi)$, let $\mu \in (0, \mu^*(\xi))$ be such that

$$c = \frac{\lambda_0(\xi, \mu, a_0)}{\mu}.$$

If (H3) holds, let $\phi(\cdot) \in X_p^+$ be the positive principal eigenfunction of $\mathcal{K}_{\xi, \mu} - I + a_0(\cdot)I$ with $\|\phi(\cdot)\|_{X_p} = 1$.

Theorem 2.1 (Existence of traveling wave solutions). *Assume (H1)-(H4). Then for any $\xi \in S^{N-1}$ and $c > c^*(\xi)$, there is a bounded measurable function $\Phi : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^+$ such that the following hold.*

- (1) $\Phi(\cdot, \cdot)$ generates a traveling wave solution connecting $u^+(\cdot)$ and 0 and propagating in the direction of ξ with speed c . Moreover, $\lim_{x \cdot \xi \rightarrow \infty} \frac{\Phi(x, z)}{e^{-\mu x \cdot \xi} \phi(x + z)} = 1$ uniformly in $z \in \mathbb{R}^N$.
- (2) Let $U(t, x; z) = u(t, x; \Phi(\cdot, z), z) (= \Phi(x - ct\xi, z + ct\xi))$. Then

$$U_t(t, x; z) > 0 \quad \forall t \in \mathbb{R}, x, z \in \mathbb{R}^N,$$

$$\lim_{x \cdot \xi - ct \rightarrow -\infty} U_t(t, x; z) = 0, \text{ and } \lim_{x \cdot \xi - ct \rightarrow \infty} \frac{U_t(t, x; z)}{e^{-\mu(x \cdot \xi - ct)} \phi(x + z)} = \mu c \text{ uniformly in } z \in \mathbb{R}^N.$$

Remark 2.2. Let $\Phi(x, z)$ be as in Theorem 2.1 and $\Psi(\eta, z) = \Phi(\eta\xi, z - \eta\xi)$. Then $U(t, x; z) = \Psi(x \cdot \xi - ct, z + x)$ and $\Psi(\eta, z)$ is differentiable in η and $\Psi_\eta(\eta, z) < 0$.

Theorem 2.2 (Uniqueness and continuity of traveling wave solutions). *Assume (H1)-(H4). Let $\Phi(\cdot, \cdot)$ be as in Theorem 2.1.*

- (1) Suppose that $\Phi_1(\cdot, \cdot)$ also generates a traveling wave solution of (1.2) in the direction of ξ with speed c and $\lim_{x \cdot \xi \rightarrow \infty} \frac{\Phi_1(x, z)}{\Phi(x, z)} = 1$ uniformly in $z \in \mathbb{R}$. Then $\Phi_1(x, z) \equiv \Phi(x, z)$.
- (2) $\Phi(x, z)$ is continuous in $(x, z) \in \mathbb{R}^N$.

Theorem 2.3 (Stability of traveling wave solutions). *Assume (H1)-(H4).*

Let $U(t, x) = U(t, x; 0) = \Phi(x - ct\xi, ct\xi)$, where $\Phi(\cdot, \cdot)$ is as in Theorem 2.1. For any $u_0 \in X^+$ satisfying that $\lim_{x \cdot \xi \rightarrow \infty} \frac{u_0(x)}{U(0, x)} = 1$ and $\liminf_{x \cdot \xi \rightarrow -\infty} u_0(x) > 0$, there holds

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \left| \frac{u(t, x; u_0, 0)}{U(t, x)} - 1 \right| = 0.$$

We remark that by the spreading property of $c^*(\xi)$, it is not difficult to see that (1.2) has no traveling wave solutions in the direction of $\xi \in S^{N-1}$ with propagating speed smaller than $c^*(\xi)$. Theorems 2.1-2.3 show the existence, uniqueness, and stability of traveling wave solutions of (1.2) in any given direction with speed greater than the spreading speed in that direction for the above mentioned three special but important cases, that is, the nonlocal dispersal is

nearly local; the periodic habitat is nearly globally homogeneous or it is nearly homogeneous in a region where it is most conducive to population growth in the zero-limit population. It remains open whether (1.2) has a traveling wave solution in the given direction of $\xi \in S^{N-1}$ with speed $c = c^*(\xi)$ for these special cases. It also remains open whether a general spatially periodic monostable equation with nonlocal dispersal in \mathbb{R}^N with $N \geq 3$ has traveling wave solutions connecting the spatially periodic positive stationary solution u^+ and 0 and propagating with constant speeds.

3 Comparison Principle and Sub- and Super-solutions

In this section, we first in 3.1 present the comparison principle for (sub-, super-) solutions of (2.5) and some related nonlocal linear evolution equations. Then we construct in 3.2 some sub- and super-solutions to be used in the proofs of the main results in later sections.

3.1 Comparison principle

Consider (2.5). For given $a(\cdot, \cdot) \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ with $a(t, \cdot) \in X_p$ for every $t \in \mathbb{R}$, consider also

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + a(t,x+z)u(t,x), \quad x \in \mathbb{R}^N. \quad (3.1)$$

Definition 3.1. A bounded Lebesgue measurable function $u(t,x)$ on $[0,T) \times \mathbb{R}^N$ is called a super-solution (or sub-solution) of (2.5) if for any $x \in \mathbb{R}^N$, $u(t,x)$ is absolutely continuous on $[0,T)$ (and so $\frac{\partial u}{\partial t}$ exists a.e on $[0,T)$) and satisfies that for each $x \in \mathbb{R}^N$,

$$\frac{\partial u}{\partial t} \geq (\text{or } \leq) \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + f(x+z,u)u(t,x)$$

for a.e. $t \in (0,T)$.

Sub and super-solutions of (3.1) are defined similarly. Throughout this subsection, we assume (H1) and (H2).

Proposition 3.1 (Comparison principle).

- (1) If $u_1(t,x)$ and $u_2(t,x)$ are sub-solution and super-solution of (3.1) on $[0,T)$, respectively, $u_1(0,\cdot) \leq u_2(0,\cdot)$, and $u_2(t,x) - u_1(t,x) \geq -\beta_0$ for $(t,x) \in [0,T) \times \mathbb{R}^N$ and some $\beta_0 > 0$, then $u_1(t,\cdot) \leq u_2(t,\cdot)$ for $t \in [0,T)$.
- (2) If $u_1(t,x)$ and $u_2(t,x)$ are bounded sub- and super-solutions of (2.5) on $[0,T)$, respectively, and $u_1(0,\cdot) \leq u_2(0,\cdot)$, then $u_1(t,\cdot) \leq u_2(t,\cdot)$ for $t \in [0,T)$.
- (3) For every $u_0 \in \tilde{X}^+$, $u(t,x;u_0,z)$ exists for all $t \geq 0$, where $u(t,x;u_0,z)$ is the solution of (2.5) with $u(0,x;u_0,z) = u_0(z)$.

- (4) Suppose that $u_1, u_2 \in \tilde{X}$, $u_1 \leq u_2$, and $\{x \in \mathbb{R}^N \mid u_2(x) > u_1(x)\}$ has positive Lebesgue measure. Then $u(t, x; u_1, z) < u(t, x; u_2, z)$ for every $t > 0$ at which both $u(t, \cdot; u_1)$ and $u(t, \cdot; u_2)$ exist and $x, z \in \mathbb{R}^N$.

Proof. It follows from the arguments in [54, Proposition 2.1] and [54, Proposition 2.2]. \square

3.2 Sub- and super-solutions

Throughout this subsection, we assume (H1)-(H4) and put $a_0(x) = f(x, 0)$.

For given $\xi \in S^{N-1}$, let $\mu^*(\xi)$ be such that

$$c^*(\xi) = \frac{\lambda_0(\xi, \mu^*(\xi), a_0)}{\mu^*(\xi)}.$$

Fix $\xi \in S^{N-1}$ and $c > c^*(\xi)$. Let $0 < \mu < \mu_1 < \min\{2\mu, \mu^*(\xi)\}$ be such that $c = \frac{\lambda_0(\xi, \mu, a_0)}{\mu}$ and $\frac{\lambda_0(\xi, \mu, a_0)}{\mu} > \frac{\lambda_0(\xi, \mu_1, a_0)}{\mu_1} > c^*(\xi)$. Let $\phi(\cdot)$ and $\phi_1(\cdot)$ be positive eigenfunctions of $\mathcal{K}_{\xi, \mu} - I + a_0(\cdot)I$ associated to $\lambda_0(\xi, \mu, a_0)$ and $\lambda_0(\xi, \mu_1, a_0)$ with $\|\phi(\cdot)\|_{X_p} = 1$ and $\|\phi_1(\cdot)\|_{X_p} = 1$, respectively. If no confusion occurs, we may write $\lambda_0(\mu, \xi, a_0)$ as $\lambda(\mu)$.

For given $d_1 > 0$, let

$$\underline{v}^1(t, x; z, T, d_1) = e^{-\mu(x \cdot \xi + cT - ct)} \phi(x + z) - d_1 e^{-\mu_1(x \cdot \xi + cT - ct)} \phi_1(x + z). \quad (3.2)$$

We may write $\underline{v}^1(t, x; z, T)$ for $\underline{v}^1(t, x; z, T, d_1)$ for fixed $d_1 > 0$ or if no confusion occurs.

Proposition 3.2. *For any $z \in \mathbb{R}^N$ and $T > 0$, $\underline{v}^1(t, x; z, T)$ is a sub-solution of (2.5) provided that d_1 is sufficiently large.*

Proof. First of all, let $\varphi = e^{-\mu(x \cdot \xi + cT - ct)} \phi(x + z)$ and $\varphi_1 = d_1 e^{-\mu_1(x \cdot \xi + cT - ct)} \phi_1(x + z)$. Let $M = \max_{x \in \mathbb{R}^N} \phi(x) (> 0)$. Let $L > 0$ be such that $-f_u(x + z, u) \leq L$ for $0 \leq u \leq M$. Let d_0 be defined by

$$d_0 = \max \left\{ \frac{\max_{x \in \mathbb{R}^N} \phi(x)}{\min_{x \in \mathbb{R}^N} \phi_1(x)}, \frac{L \max_{x \in \mathbb{R}^N} \phi^2(x)}{(\mu_1 c - \lambda(\mu_1)) \min_{x \in \mathbb{R}^N} \phi_1(x)} \right\}$$

Fix $z \in \mathbb{R}^N$ and $T > 0$. We prove that $\underline{v}^1(t, x; z, T)$ is a sub-solution of (2.5) for $d_1 \geq d_0$, that is, for any $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,

$$\frac{\partial \underline{v}^1}{\partial t} - \left[\int_{\mathbb{R}^N} k(y - x) \underline{v}^1(t, y; z, T) dy - \underline{v}^1(t, x; z, T) + f(x + z, \underline{v}^1(t, x; z, T)) \underline{v}^1(t, x; z, T) \right] \leq 0. \quad (3.3)$$

First, for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $\underline{v}^1(t, x; z, T) \leq 0$, $f(x + z, \underline{v}^1(t, x; z, T)) = f(x + z, 0)$. Hence

$$\begin{aligned} & \frac{\partial \underline{v}^1}{\partial t} - \left[\int_{\mathbb{R}^N} k(y - x) \underline{v}^1(t, y; z, T) dy - \underline{v}^1(t, x; z, T) + f(x + z, \underline{v}^1(t, x; z, T)) \underline{v}^1(t, x; z, T) \right] \\ &= -(\mu_1 c - \lambda(\mu_1)) \varphi_1 \leq 0. \end{aligned}$$

Therefore (3.3) holds for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $\underline{v}^1(t, x; z, T) \leq 0$.

Next, consider $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $\underline{v}^1(t, x; z, T) > 0$. By $d_1 \geq d_0$, we must have $x \cdot \xi + cT - ct \geq 0$. Then $\underline{v}^1(t, x; z, T) \leq e^{-\mu(x \cdot \xi + cT - ct)} \phi(x + z) \leq \phi(x + z) \leq M$. Note that for $0 < y < M$,

$$\begin{aligned} -(\mu_1 c - \lambda(\mu_1)) - f_u(x + z, y) \frac{(\varphi)^2}{\varphi_1} &\leq -(\mu_1 c - \lambda(\mu_1)) + L \frac{(\varphi)^2}{\varphi_1} \\ &= -(\mu_1 c - \lambda(\mu_1)) + \frac{L \phi^2(x + z)}{d_1 \phi_1(x + z)} e^{(\mu_1 - 2\mu)(x \cdot \xi + cT - ct)} \\ &\leq -(\mu_1 c - \lambda(\mu_1)) + \frac{L \max_{y \in \mathbb{R}^N} \phi^2(y)}{d_1 \max_{y \in \mathbb{R}^N} \phi_1(y)} \\ &\leq 0. \end{aligned}$$

Therefore, for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $\underline{v}^1(t, x; z, T) > 0$,

$$\begin{aligned} &\frac{\partial \underline{v}^1}{\partial t} - \left[\int_{\mathbb{R}^N} k(y - x) \underline{v}^1(t, y; z, T) dy - \underline{v}^1(t, x; z, T) + f(x + z, \underline{v}^1) \underline{v}^1(t, x; z, T) \right] \\ &= \mu c \varphi - \mu_1 c \varphi_1 - \left[\int_{\mathbb{R}^N} k(y - x) \underline{v}^1(t, y; z, T) dy - \underline{v}^1(t, x; z, T) + f(x + z, \underline{v}^1) \underline{v}^1(t, x; z, T) \right] \\ &= (\mu c - \lambda(\mu)) \varphi - (\mu_1 c - \lambda(\mu_1)) \varphi_1 + f(x + z, 0) \underline{v}^1(t, x; z, T) - f(x + z, \underline{v}^1) \underline{v}^1(t, x; z, T) \\ &= -(\mu_1 c - \lambda(\mu_1)) \varphi_1 - f_u(x + z, y) (\varphi - \varphi_1)^2 \quad (\text{for some } y \in (0, M)) \\ &\leq -(\mu_1 c - \lambda(\mu_1)) \varphi_1 - f_u(x + z, y) (\varphi)^2 \\ &= [-(\mu_1 c - \lambda(\mu_1)) - f_u(x + z, y) \frac{(\varphi)^2}{\varphi_1}] \varphi_1 \\ &\leq 0. \end{aligned}$$

Hence (3.3) also holds for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $\underline{v}^1(t, x; z, T) > 0$. The proposition then follows. \square

Proposition 3.3. *Let ϕ_0 be the positive principal eigenfunction of $\mathcal{K} - I + a_0(\cdot)I$ with $\|\phi_0\|_{X_p} = 1$. Then for any $z \in \mathbb{R}^N$ and $0 < b \ll 1$, $\underline{v}^2(t, x; z, b) := b\phi_0(x + z)$ is a sub-solution of (2.5).*

Proof. Fix $z \in \mathbb{R}^N$. Observe that

$$\int_{\mathbb{R}^N} k(y - x) \phi_0(y + z) dy - \phi_0(x + z) + f(x + z, 0) \phi_0(x + z) = \lambda_0 \phi_0(x + z) \quad \forall x \in \mathbb{R}^N.$$

Observe also that $\max_{x \in \mathbb{R}^N} \lambda_0 \phi_0(x + z) > 0$ and then

$$\lambda_0 b \phi_0(x + z) \geq (f(x + z, 0) - f(x + z, b\phi_0(x + z))) b \phi_0(x + z) \quad \forall 0 < b \ll 1.$$

It then follows that

$$\int_{\mathbb{R}^N} k(y - x) b \phi_0(y + z) dy - b \phi_0(x + z) + f(x + z, b \phi_0(x + z)) b \phi_0(x + z) \geq 0 \quad \forall x \in \mathbb{R}^N, 0 < b \ll 1.$$

Hence $\underline{v}^2(t, x; z, b)$ is a sub-solution of (2.5) for $0 < b \ll 1$. \square

For given $0 < b \ll 1$, there is $M > 0$ such that for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $M - 2\delta_0 \leq x \cdot \xi + cT - ct \leq M$ (δ_0 is the nonlocal dispersal distance in (1.2)),

$$\underline{v}^1(t, x; z, T) \geq b. \quad (3.4)$$

Proposition 3.4. *Let $0 < b \ll 1$ and $M > 0$ be such that (3.4) holds and $z \in \mathbb{R}^N$, $T > 0$. Let*

$$\underline{u}(t, x; z, T, d_1, b) = \begin{cases} \max\{b\phi_0(x+z), \underline{v}^1(t, x; z, T, d_1)\} & \text{for } x \cdot \xi + cT - ct < M \\ \underline{v}^1(t, x; z, T, d_1) & \text{for } x \cdot \xi + cT - ct \geq M. \end{cases}$$

Then $\underline{u}(t, x; z, T, d_1, b)$ is a sub-solution of (2.5).

Proof. First, it is not difficult to see that for any $x, z \in \mathbb{R}^N$, there are at most two ts such that $b\phi_0(x+z) = \underline{v}^1(t, x; z, T)$. Hence for any fixed $x, z \in \mathbb{R}^N$, $\underline{u}(t, x; z, T) (= \underline{u}(t, x; z, t, b, d_1))$ is continuous at every t and is differentiable in t for a.e. t . Moreover, for any t at which $\underline{u}(t, x; z, T)$ is differentiable, there holds

$$\frac{\partial \underline{u}(t, x; z, T)}{\partial t} \leq \int_{\mathbb{R}^N} k(y-x) \underline{u}(t, y; z, T) dy - \underline{u}(t, x; z, T) + \underline{u}(t, x; z, T) f(x+z, \underline{u}(t, x; z, T)).$$

Therefore, $\underline{u}(t, x; z, T)$ is a sub-solution of (2.5). \square

For given $d_2 \geq 0$, let

$$\bar{v}(t, x; z, T, d_2) = e^{-\mu(x \cdot \xi + cT - ct)} \phi(x+z) + d_2 e^{-\mu_1(x \cdot \xi + cT - ct)} \phi_1(x+z)$$

and

$$\bar{u}(t, x; z, T, d_2) = \min\{\bar{v}(t, x; z, T, d_2), u^+(x+z)\}.$$

We may write $\bar{v}(t, x; z, T)$ and $\bar{u}(t, x; z, T)$ for $\bar{v}(t, x; z, T, d_2)$ and $\bar{u}(t, x; z, T, d_2)$, respectively, if no confusion occurs.

Proposition 3.5. *For any $d_2 \geq 0$, $z \in \mathbb{R}^N$, and $T > 0$, $\bar{u}(t, x; z, T)$ is a super-solution of (2.5).*

Proof. It suffices to prove that $\bar{v}(t, x; z, T)$ is a super-solution.

Let $\varphi_2 = d_2 e^{-\mu_1(x \cdot \xi + cT - ct)} \phi_1(x+z)$. By direct calculation, we have

$$\begin{aligned} & \frac{\partial \bar{v}}{\partial t} - \left[\int_{\mathbb{R}^N} k(y-x) \bar{v}(t, y; z, T) dy - \bar{v}(t, x; z, T) + f(x+z, \bar{v}) \bar{v}(t, x; z, T) \right] \\ & \geq \frac{\partial \bar{v}}{\partial t} - \left[\int_{\mathbb{R}^N} k(y-x) \bar{v}(t, y; z, T) dy - \bar{v}(t, x; z, T) + f(x+z, 0) \bar{v}(t, x; z, T) \right] \\ & = (\mu_1 c - \lambda(\mu_1)) \varphi_2 \\ & \geq 0. \end{aligned}$$

The proposition thus follows. \square

In the rest of this section, we fix $d_1^* \gg 1$, $d_2^* \geq 0$, and $0 < b^* \ll 1$. Let

$$u_{0,z,T}^-(x) = \underline{u}(0, x; z, T, d_1^*, b^*) \quad \text{and} \quad u_{0,z,T}^+(x) = \bar{u}(0, x; z, T, d_2^*). \quad (3.5)$$

Then by Proposition 3.4,

$$\begin{aligned} u(t, x; u_{0,z,T}^-, z) &\geq \underline{u}(t, x; z, T) \\ &= \underline{u}(0, x; z, T - t) \\ &= u_{0,z,T-t}^-(x). \end{aligned}$$

Similarly,

$$u(t, x; u_{0,z,T}^+, z) \leq u_{0,z,T-t}^+(x).$$

Proposition 3.6. *For any given $z \in \mathbb{R}^N$, the following hold:*

(1) *For any $t_2 > t_1 > 0$,*

$$u(t_2 + t, x; u_{0,z,t_2}^-, z) \geq u(t_1 + t, x; u_{0,z,t_1}^-, z) \quad \forall t > -t_1, \quad x \in \mathbb{R}^N;$$

(2)

$$u(t_2 + t, x; u_{0,z,t_2}^+, z) \leq u(t_1 + t, x; u_{0,z,t_1}^+, z) \quad \forall t > -t_1, \quad x \in \mathbb{R}^N.$$

Proof. (1) For given $z \in \mathbb{R}^N$ and $t_2 > t_1 > 0$, by Proposition 3.4,

$$\begin{aligned} u(t_2 - t_1, x; u_{0,z,t_2}^-, z) &\geq \underline{u}(t_2 - t_1, x; z, t_2) \\ &= u_{0,z,t_2-(t_2-t_1)}^-(x) \\ &= u_{0,z,t_1}^-(x). \end{aligned}$$

Hence

$$\begin{aligned} u(t_2 + t, x; u_{0,z,t_2}^-, z) &= u(t_1 + t, x; u(t_2 - t_1, \cdot; u_{0,z,t_2}^-, z), z) \\ &\geq u(t_1 + t, x; u_{0,z,t_1}^-, z). \end{aligned}$$

(1) is thus proved.

(2) It follows by the similar arguments in (1) and Proposition 3.5. □

4 Existence of Traveling Wave Solutions and Proof of Theorem 2.1

In this section, we investigate the existence of traveling wave solutions of (1.2) and prove Theorem 2.1. Throughout this section, we assume (H1)-(H4).

Let $u_{0,z,T}^\pm$ be as in (3.5). Let

$$\Phi^\pm(x, z) = \lim_{\tau \rightarrow \infty} u(\tau, x; u_{0,z,\tau}^\pm, z) \quad (4.1)$$

and

$$U^\pm(t, x; z) = \lim_{\tau \rightarrow \infty} u(t + \tau, x; u_{0,z,\tau}^\pm, z). \quad (4.2)$$

By Proposition 3.6, the limits in the above exist for all $t \in \mathbb{R}$ and $x, z \in \mathbb{R}^N$. Moreover, it is easy to see that $\Phi^-(x, z)$ is lower semi-continuous in $(x, z) \in \mathbb{R}^N \times \mathbb{R}^N$ and $\Phi^+(x, z)$ is upper semi-continuous.

We will show that $u = U^+(t, x; 0)$ and $u = U^-(t, x; 0)$ are traveling wave solutions of (1.2) in the direction of ξ with speed c generated by $\Phi^+(\cdot, \cdot)$ and $\Phi^-(\cdot, \cdot)$, respectively, and that $\Phi(\cdot, \cdot) := \Phi^+(\cdot, \cdot)$ satisfies Theorem 2.1(1)-(2).

To this end, we first prove some lemmas.

Lemma 4.1. *For each $z \in \mathbb{R}^N$, $u(t, x) = U^\pm(t, x; z)$ are entire solutions of (2.5).*

Proof. We prove the case that $u(t, x) = U^+(t, x; z)$. The other case can be proved similarly.

Fix $z \in \mathbb{R}^N$. Observe that for any $x \in \mathbb{R}^N$,

$$\begin{aligned} u(t + \tau, x; u_{0,z,\tau}^+, z) &= u(\tau, x; u_{0,z,\tau}^+, z) + \int_0^t \int_{\mathbb{R}^N} k(y - x) u(s + \tau, y; u_{0,z,\tau}^+, z) dy ds \\ &\quad + \int_0^t [-u(s + \tau, x; u_{0,z,\tau}^+, z) + u(s + \tau, x; u_{0,z,\tau}^+, z) f(x + z, u(s + \tau, x; u_{0,z,\tau}^+, z))] ds \end{aligned}$$

Letting $\tau \rightarrow \infty$, we have

$$u(t, x) = u(0, x) + \int_0^t \left[\int_{\mathbb{R}^N} k(y - x) u(s, y) dy - u(s, x) + u(s, x) f(x + z, u(s, x)) \right] ds.$$

This implies that $u(t, x)$ is differentiable in t and satisfies (2.5) for all $t \in \mathbb{R}$. \square

Observe that

$$U^\pm(t, x; z) = u(t, x; \Phi^\pm(\cdot, z), z) \quad \forall t \in \mathbb{R}, \quad x, z \in \mathbb{R}^N.$$

Lemma 4.2. $u(t, x; \Phi^\pm(\cdot, z), z) = \Phi^\pm(x - ct\xi, z + ct\xi)$, $\lim_{x \cdot \xi \rightarrow -\infty} (\Phi^\pm(x, z) - u^+(x + z)) = 0$ and

$$\lim_{x \cdot \xi \rightarrow \infty} \frac{\Phi^\pm(x, z)}{e^{-\mu x \cdot \xi} \phi(x + z)} = 1 \text{ uniformly in } z \in \mathbb{R}^N.$$

Proof. We prove the lemma for $\Phi^+(\cdot, \cdot)$. It can be proved similarly for $\Phi^-(\cdot, \cdot)$.

First of all, we have

$$\begin{aligned} u(t, x; \Phi^+(\cdot, z), z) &= \lim_{\tau \rightarrow \infty} u(t, x; u(\tau, x; u_{0,z,\tau}^+, z), z) \\ &= \lim_{\tau \rightarrow \infty} u(t + \tau, x; u_{0,z,\tau}^+, z) \\ &= \lim_{\tau \rightarrow \infty} u(t + \tau, x - ct\xi; u_{0,z+ct\xi,t+\tau}^+, z + ct\xi) \\ &= \Phi^+(x - ct\xi, z + ct\xi). \end{aligned}$$

Note that

$$\begin{aligned}
\underline{u}(t+T, x; z, T) &= e^{-\mu(x \cdot \xi - ct)} \phi(x+z) - d_1 e^{-\mu_1(x \cdot \xi - ct)} \phi_1(x+z) \\
&\leq u(t, x; \Phi^+(\cdot, z), z) \\
&\leq \bar{u}(t+T, x; z, T) \\
&= e^{-\mu(x \cdot \xi - ct)} \phi(x+z) + d_2 e^{-\mu_1(x \cdot \xi - ct)} \phi_1(x+z)
\end{aligned}$$

for $t \in \mathbb{R}$ and $x, z \in \mathbb{R}^N$. Thus $\lim_{x \cdot \xi - ct \rightarrow \infty} \frac{\Phi^+(x - ct\xi, z + ct\xi)}{e^{-\mu(x \cdot \xi - ct)} \phi(x+z)} = 1$, which is equivalent to $\lim_{x \cdot \xi \rightarrow \infty} \frac{\Phi^+(x, z)}{e^{-\mu x \cdot \xi} \phi(x+z)} = 1$, uniformly in $z \in \mathbb{R}^N$.

We now prove that $\lim_{x \cdot \xi \rightarrow -\infty} (\Phi^+(x, z) - u^+(x+z)) = 0$ uniformly in $z \in \mathbb{R}^N$. Observe that there is $M > 0$ such that

$$U^+(t, x, z) \geq U^-(t, x, z) \geq b\phi_0(x+z) \quad \text{for } x \cdot \xi - ct \leq M, \quad z \in \mathbb{R}^N.$$

By Proposition 2.1, for any $\epsilon > 0$, there are $T > 0$ and $\eta^* \in \mathbb{R}$ such that

$$|U^+(T, x, z) - u^+(x+z)| < \epsilon \quad \text{for } x \cdot \xi \leq \eta^*, \quad z \in \mathbb{R}^N.$$

This implies that

$$|\Phi^+(x, z) - u^+(x+z)| \leq \epsilon \quad \text{for } x \cdot \xi \leq \eta^* + cT, \quad z \in \mathbb{R}^N$$

and hence $\lim_{x \cdot \xi \rightarrow -\infty} (\Phi^+(x, z) - u^+(x+z)) = 0$ uniformly in $z \in \mathbb{R}^N$. \square

Corollary 4.1. *Both $\Phi^+(\cdot, \cdot)$ and $\Phi^-(\cdot, \cdot)$ generate traveling wave solutions of (1.2) in the direction of ξ with speed c .*

Proof. First of all, by Lemmas 4.1 and 4.2, both $\Phi^+(\cdot, \cdot)$ and $\Phi^-(\cdot, \cdot)$ satisfy (2.7) and (2.8).

Next, for any $x, x' \in \mathbb{R}^N$ with $x \cdot \xi = x' \cdot \xi$, $z \in \mathbb{R}^N$, and $\tau \in \mathbb{R}$, we have

$$\begin{aligned}
u(\tau, x'; u_{0, z-x', \tau}^\pm(\cdot), z-x') &= u(\tau, x; u_{0, z-x', \tau}^\pm(\cdot + x' - x), z-x' + (x' - x)) \\
&= u(\tau, x; u_{0, z-x, \tau}^\pm(\cdot), z-x).
\end{aligned}$$

This implies that $\Phi^\pm(\cdot, \cdot)$ satisfies (2.9).

Observe now that $u_{0, z+p_i \mathbf{e}_i, \tau}^\pm = u_{0, z, \tau}^\pm$ for any $\tau \in \mathbb{R}$ and $z \in \mathbb{R}^N$. It then follows that $\Phi^\pm(x, z + p_i \mathbf{e}_i) = \Phi^\pm(x, z)$ and hence $\Phi^\pm(\cdot, \cdot)$ satisfies (2.10).

Therefore, both $\Phi^+(\cdot, \cdot)$ and $\Phi^-(\cdot, \cdot)$ generate traveling wave solutions of (1.2) in the direction of ξ with speed c . \square

Lemma 4.3. $\lim_{x \cdot \xi - ct \rightarrow -\infty} U_t^\pm(t, x; z) = 0$ uniformly in $z \in \mathbb{R}^N$.

Proof. Note that

$$\begin{aligned} U_t^\pm(t, x; z) &= \int_{\mathbb{R}^N} k(y-x) U^\pm(t, y; z) dy - U^\pm(t, x; z) + U^\pm(t, x; z) f(x+z, U^\pm(t, x; z)) \\ &= \int_{\|y\| \leq \delta_0} k(y) U^\pm(t, x+y; z) dy - U^\pm(t, x; z) + U^\pm(t, x; z) f(x+z, U^\pm(t, x; z)). \end{aligned}$$

Note also that

$$\lim_{x \cdot \xi - ct \rightarrow -\infty} (U^\pm(t, x; z) - u^\pm(x+z)) = 0$$

uniformly in $z \in \mathbb{R}^N$. It then follows that

$$\begin{aligned} \lim_{x \cdot \xi - ct \rightarrow -\infty} U_t^\pm(t, x; z) &= \lim_{x \cdot \xi - ct \rightarrow -\infty} \left[U_t^\pm(t, x; z) - \int_{\mathbb{R}^N} k(y) u^\pm(y+x+z) dy + u^\pm(x+z) \right. \\ &\quad \left. - u^\pm(x+z) f(x+z, u^\pm(x+z)) \right] \\ &= \lim_{x \cdot \xi - ct \rightarrow -\infty} \left[\int_{\mathbb{R}^N} k(y) (U^\pm(t, x+y; z) - u^\pm(x+y+z)) dy \right. \\ &\quad \left. - (U^\pm(t, x; z) - u^\pm(x+z)) \right. \\ &\quad \left. + (U^\pm(t, x; z) f(x+z, U^\pm(t, x; z)) - u^\pm(x+z) f(x+z, u^\pm(x+z))) \right] \\ &= 0 \quad \text{uniformly in } z \in \mathbb{R}^N. \end{aligned}$$

□

Lemma 4.4. $\lim_{x \cdot \xi - ct \rightarrow \infty} \frac{U_t^\pm(t, x; z)}{e^{-\mu(x \cdot \xi - ct)} \phi(x+z)} = \mu c$ uniformly in $z \in \mathbb{R}^N$.

Proof. We prove the lemma for $U^+(t, x; z)$. It can be proved similarly for $U^-(t, x; z)$.

First, let $U(t, x; z) = U^+(t, x; z)$. By Lemma 4.2, for any $\epsilon > 0$, there is $M > 0$ such that for any $x, z \in \mathbb{R}^N$ and $t \in \mathbb{R}$ with $x \cdot \xi - ct \geq M$,

$$\left| \frac{U(t, x; z)}{e^{-\mu(x \cdot \xi - ct)}} - \phi(x+z) \right| < \epsilon \quad (4.3)$$

and

$$|f(x+z, U(t, x; z)) - f(x+z, 0)| < \epsilon. \quad (4.4)$$

Observe that

$$\mu c \phi(x+z) = \int_{\mathbb{R}^N} e^{-\mu(y-x) \cdot \xi} k(y-x) \phi(y+z) dy - \phi(x+z) + a_0(x+z) \phi(x+z) \quad (4.5)$$

for all $x, z \in \mathbb{R}^N$, where $a_0(x+z) = f(x+z, 0)$, and

$$U_t(t, x; z) = \int_{\mathbb{R}^N} k(y-x) U(t, y; z) dy - U(t, x; z) + U(t, x; z) f(x+z, U(t, x; z)) \quad (4.6)$$

for all $t \in \mathbb{R}$ and $x, z \in \mathbb{R}^N$. By (4.3)-(4.6), we have

$$\begin{aligned}
\left| \frac{U_t(t, x; z)}{e^{-\mu(x \cdot \xi - ct)} \phi(x + z)} - \mu c \right| &= \frac{1}{\phi(x + z)} \left| \int_{\mathbb{R}^N} e^{-\mu(y-x) \cdot \xi} k(y-x) \left(\frac{U(t, y; z)}{e^{-\mu(y \cdot \xi - ct)}} - \phi(y + z) \right) dy \right. \\
&\quad - \left(\frac{U(t, x; z)}{e^{-\mu(x \cdot \xi - ct)}} - \phi(x + z) \right) \\
&\quad + \left(\frac{U(t, x; z)}{e^{-\mu(x \cdot \xi - ct)}} - \phi(x + z) \right) f(x + z, U(t, x; z)) \\
&\quad \left. + \phi(x + z) (f(x + z, U(t, x; z)) - f(x + z, 0)) \right| \\
&\leq \epsilon \left[\int_{\mathbb{R}^N} e^{-\mu(y-x) \cdot \xi} k(y-x) dy \right. \\
&\quad \left. + 1 + |f(x + z, U(t, x; z))| + \phi(x + z) \right]
\end{aligned}$$

for all $x, z \in \mathbb{R}^N$ and $t \in \mathbb{R}$ with $x \cdot \xi - ct \geq M + \delta_0$, where δ_0 is the nonlocal dispersal distance in (1.2). It then follows that

$$\lim_{x \cdot \xi - ct \rightarrow \infty} \frac{U_t^\pm(t, x; z)}{e^{-\mu(x \cdot \xi - ct)} \phi(x + z)} = \mu c$$

uniformly in $z \in \mathbb{R}^N$. □

Proof of Theorem 2.1. Let $\Phi(x, z) = \Phi^+(x, z)$ and $U(t, x; z) = U^+(t, x; z)$. Note that $U(t, x; z) = u(t, x; \Phi(\cdot, z), z)$. We show that $\Phi(\cdot, \cdot)$ and $U(\cdot, \cdot; \cdot)$ satisfy Theorem 2.1(1) and (2), respectively.

(1) It follows from Corollary 4.1 and Lemma 4.2.

(2) By Lemmas 4.3 and 4.4, we only need to prove that $U_t(t, x; z) > 0$ for all $(t, x, z) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$.

For any $t_1 < t_2$, we have

$$u_{0,z,t_1}^+(x) \geq u_{0,z,t_2}^+(x) \quad \forall x, z \in \mathbb{R}^N.$$

Hence

$$\begin{aligned}
u(t_1, x; \Phi^+(\cdot, z), z) &= u(t_2 + t_1 - t_2, x; \Phi^+(\cdot, z), z) \\
&= \lim_{n \rightarrow \infty} u(t_2, x; u(n + t_1 - t_2, \cdot; u_{0,z,n}^+, z), z) \\
&\leq \lim_{n \rightarrow \infty} u(t_2, x; u(n + t_1 - t_2, \cdot; u_{0,z,n+t_1-t_2}^+, z), z) \\
&= u(t_2, x; \Phi^+(\cdot, z), z).
\end{aligned}$$

Therefore, $U(t, x; z) = u(t, x; \Phi^+(\cdot, z), z)$ is nondecreasing as t increases.

Let $v(t, x; z) = u_t(t, x; \Phi^+(\cdot, z), z)$. Then $v(t, x; z) \geq 0$. By Lemma 4.4, for any $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$, the set $\{x \in \mathbb{R}^N \mid v(t, x; z) > 0\}$ has positive Lebesgue measure. Note that $v(t, x; z)$ satisfies

$$v_t(t, x; z) = \int_{\mathbb{R}} k_\delta(y-x) v(t, y; z) dy - v(t, x; z) + a(t, x; z) v(t, x; z) \quad (4.7)$$

where $a(t, x; z) = f(x + z, u(t, x; \Phi^+(\cdot, z), z)) + u(t, x; \Phi^+(\cdot, z), z)f_u(x + z, u(t, x; \Phi^+(\cdot, z), z))$. Then by Proposition 3.1, we have

$$v(t, x; z) > 0 \quad \forall t \in \mathbb{R}, x, z \in \mathbb{R}^N.$$

This implies that $U_t(t, x; z) > 0$ for all $t \in \mathbb{R}$ and $x, z \in \mathbb{R}^N$. \square

5 Uniqueness and Continuity of Traveling Wave Solutions and Proof of Theorem 2.2

In this section, we investigate the uniqueness and continuity of traveling wave solutions of (1.2) and prove Theorem 2.2 by the “squeezing” techniques developed in [8] and [22].

Throughout this section, we fix $\xi \in S^{N-1}$ and $c > c^*(\xi)$. Let μ^* be such that

$$c^*(\xi) = \frac{\lambda_0(\mu^*, \xi, a_0)}{\mu^*} < \frac{\lambda_0(\tilde{\mu}, \xi, a_0)}{\tilde{\mu}} \quad \forall \tilde{\mu} \in (0, \mu^*).$$

We fix $c > c^*(\xi)$ and $\mu \in (0, \mu^*)$ with $\frac{\lambda_0(\mu, \xi, a_0)}{\mu} = c$ and assume that $U^\pm(t, x; z)$ and $\Phi^\pm(x, z)$ are as in section 4. We put $\Phi(x, z) = \Phi^+(x, z)$ and $U(t, x; z) = U^+(t, x; z)$. Let $U_1(t, x; z) = u(t, x; \Phi_1(\cdot, z), z) (\equiv \Phi_1(x - ct\xi, z + ct\xi))$.

We first prove some lemmas, some of which will also be used in next section. By Lemmas 4.2 and 4.4, there is $M_0 > 0$ such that

$$0 < \sup_{x \cdot \xi - ct \geq M_0, z \in \mathbb{R}^N} \frac{U(t, x; z)}{U_t(t, x; z)} < \infty. \quad (5.1)$$

Observe that there is $\sigma_0 > 0$ such that

$$U(t, x; z) \geq \sigma_0 \quad \text{for } x \cdot \xi - ct \leq M_0. \quad (5.2)$$

Let

$$\eta_0 = \inf_{0 < u \leq 2u_{\sup}^+} (-f_u(x, u))\sigma_0, \quad (5.3)$$

where $u_{\sup}^+ = \sup_{x \in \mathbb{R}^N} u^+(x)$. Throughout the rest of this section, M_0, σ_0, η_0 are fixed and satisfy (5.1)-(5.3).

Lemma 5.1. *Let $\epsilon_0 \in (0, 1)$ and $\eta \in (0, (1 - \epsilon_0)\eta_0)$. There is $l > 0$ such that for each $\epsilon \in (0, \epsilon_0)$,*

$$H^\pm(t, x; z) = (1 \pm \epsilon e^{-\eta t})U(t \mp l\epsilon e^{-\eta t}, x; z), \forall t \geq 0, x, z \in \mathbb{R}^N$$

are super-/sub-solution of (2.5).

Proof. First we prove that $H^+(t, x; z)$ is a super-solution of (2.5). Let $h = \epsilon e^{-\eta t}$ and $\tau = t - l\epsilon e^{-\eta t}$. Then

$$H^+(t, x; z) = (1 + h)U(\tau, x; z), \forall t \geq 0, x, z \in \mathbb{R}^N.$$

By direct calculation, we have

$$\begin{aligned}
& \frac{\partial H^+(t, x; z)}{\partial t} - \left[\int_{\mathbb{R}^N} k(y - x) H^+(t, y; z) dy - H^+(t, x; z) + H^+(t, x; z) f(x + z, H^+(t, x; z)) \right] \\
&= -\eta h U(\tau, x; z) + (1 + l\eta h) [(\mathcal{K} - I) H^+ + f(x + z, U) H^+] - [(\mathcal{K} - I) H^+ + f(x + z, H) H^+] \\
&= -\eta h U(\tau, x; z) + l\eta h [(\mathcal{K} - I) H^+ + f(x + z, U) H^+] + [f(x + z, U) - f(x + z, H)] H^+ \\
&= -\eta h U(\tau, x; z) + l\eta h (1 + h) U_t(\tau, x; z) + [f(x + z, U) - f(x + z, H^+)] (1 + h) U(\tau, x; z) \\
&= h\eta U(\tau, x; z) [-1 + l(1 + h) \frac{U_t(\tau, x + z)}{U(\tau, x + z)} - f_u(x + z, u^*(\tau, x; z)) (1 + h) U(\tau, x; z) / \eta],
\end{aligned}$$

where $u^*(\tau, x; z)$ is some number between $U(\tau, x; z)$ and $H^+(t, x; z)$. We only need to prove that

$$-1 + l(1 + h) \frac{U_t(\tau, x; z)}{U(\tau, x; z)} - f_u(x + z, U^*(\tau, x; z)) (1 + h) U(\tau, x; z) / \eta \geq 0 \quad (5.4)$$

for all $t \geq 0$ and $x, z \in \mathbb{R}^N$.

If $t \geq 0$ and $x \in \mathbb{R}^N$ are such that $x \cdot \xi - c\tau \leq M_0$, by (5.2), (5.3), and the fact that $U_t(\tau, x; z) > 0$, (5.4) holds.

If $t \geq 0$ and $x \in \mathbb{R}^N$ are such that $x \cdot \xi - c\tau \geq M_0$, and $l \geq \sup_{x \cdot \xi - c\tau \geq M_0} \frac{U(\tau, x; z)}{U_t(\tau, x; z)}$, then (5.4) also holds.

By the similar arguments above, we can prove that $H^-(t, x; z)$ is a sub-solution of (2.5). This completes the proof. \square

Lemma 5.2. *Let $\epsilon_0 \in (0, 1)$ be given and l be as in Lemma 5.1. For any given $0 < \epsilon_1 \leq \epsilon_0$, there exists constant $M_1(\epsilon_1) > 0$ such that for all $\epsilon \in (0, \epsilon_1]$*

$$(1 - \epsilon)U(t + 3l\epsilon, x; z) \leq U(t, x; z) \leq (1 + \epsilon)U(t - 3l\epsilon, x; z) \quad \forall t \in \mathbb{R}, \quad x, z \in \mathbb{R}^N, \quad x - ct \leq -M_1(\epsilon_1).$$

Proof. Let $h(s) = (1 + s)U(t - 3ls, x; z)$. Then, $h'(s) = U(t - 3ls, x; z) - 3lU_t(t - 3ls, x; z)$. By Lemma 4.3, there exists a $M(\epsilon_1) > 0$ such that $h'(s) > 0$ for $s \in [-\epsilon_1, \epsilon_1]$, $x - ct \leq -M_1(\epsilon_1)$, and $z \in \mathbb{R}^N$. Hence, the lemma follows. \square

Lemma 5.3. *For any $\epsilon > 0$, there exists a constant $C(\epsilon) \geq 1$ such that*

$$U_1(t - 2\epsilon, x; z) \leq U(t, x; z) \leq U_1(t + 2\epsilon, x; z) \quad \forall t \in \mathbb{R}, \quad x, z \in \mathbb{R}^N, \quad x \cdot \xi - ct \geq C(\epsilon).$$

Proof. It follows from the fact that

$$\begin{aligned}
\lim_{x \cdot \xi - ct \rightarrow \infty} \frac{U_1(t, x; z)}{e^{-\mu(x \cdot \xi - ct)} \phi(x + z)} &= \lim_{x \cdot \xi - ct \rightarrow \infty} \frac{U_1(t, x; z)}{U(t, x; z)} \frac{U(t, x; z)}{e^{-\mu(x \cdot \xi - ct)} \phi(x + z)} \\
&= \lim_{x \cdot \xi - ct \rightarrow \infty} \frac{U(t, x; z)}{e^{-\mu(x \cdot \xi - ct)} \phi(x + z)} \\
&= 1
\end{aligned}$$

uniformly in $z \in \mathbb{R}^N$. \square

Lemma 5.4. Let $\epsilon_0 \in (0, 1)$ and η_0, l be as in Lemma 5.1. For any given $\epsilon \in (0, \epsilon_0)$, there is $\tau > 0$ such that

$$(1 - \epsilon e^{-\eta t})U(t - \tau + l\epsilon e^{-\eta t}, x) \leq U_1(t, x; z) \leq (1 + \epsilon e^{-\eta t})U(t + \tau - l\epsilon e^{-\eta t}, x; z)$$

for all $x, z \in \mathbb{R}^N$ and $t \geq 0$.

Proof. First by Propositions 2.1 and 3.1,

$$0 < U(t, x; z) < u^+(x + z) \quad \text{and} \quad 0 < U_1(t, x; z) < u^+(x + z) \quad \forall t \in \mathbb{R}, x, z \in \mathbb{R}^N.$$

Then by Lemma 5.3, there exists a constant $C(1)$ such that

$$U_1(t, x; z) \geq U(t - 2, x; z) \quad \forall t \in \mathbb{R}, x, z \in \mathbb{R}^N, x \cdot \xi - ct \geq C(1).$$

By (2.8), there is $t_1 \geq 2$ such that

$$U_1(t, x; z) \geq (1 - \epsilon)U(t - t_1, x; z) \quad \forall t \in \mathbb{R}, x, z \in \mathbb{R}^N, x \cdot \xi - ct < C(1).$$

Thus

$$U_1(0, x; z) \geq (1 - \epsilon)U(-t_1, x; z) = (1 - \epsilon)U(-(t_1 + l\epsilon) + l\epsilon, x; z) \quad \forall x, z \in \mathbb{R}^N.$$

It then follows Lemma 5.1 that

$$U_1(t, x; z) \geq (1 - \epsilon e^{-\eta t})U(t - (t_1 + l\epsilon) + l\epsilon e^{-\eta t}, x; z) \quad \forall t \geq 0, x, z \in \mathbb{R}^N.$$

Similarly, it can be proved that there is $t_2 \geq 2$ such that

$$U_1(t, x; z) \leq (1 + \epsilon e^{-\eta t})U(t + t_2 + l\epsilon - l\epsilon e^{-\eta t}, x; z) \quad \forall t \geq 0, x, z \in \mathbb{R}^N.$$

The lemma then follows with $\tau = \max\{t_1 + l\epsilon, t_2 + l\epsilon\}$. □

Lemma 5.5. Let $\tau > 0, t_1 > 0$, and $M \in \mathbb{R}$ be given. Suppose that $W^\pm(t, x; t_1, z)$ are the solution of (2.5) with initial

$$W^\pm(0, x; t_1, z) = U(t_1 \pm \tau, x; z)\varsigma(x - ct_1 - M) + U(t_1 \pm 2\tau, x; z)(1 - \varsigma(x - ct_1 - M)),$$

where $\varsigma(s) = 0$ for $s \leq 0$ and $\varsigma(s) = 1$ for $s > 0$. Then

$$W^+(1, x; t_1, z) \leq (1 + \epsilon)U(t_1 + 1 + 2\tau - 3l\epsilon, x; z)$$

and

$$W^-(1, x; t_1, z) \geq (1 - \epsilon)U(t_1 + 1 - 2\tau + 3l\epsilon, x; z)$$

for all $x, z \in \mathbb{R}^N$ with $x - c(1 + t_1) \leq M$ provided that $0 < \epsilon \ll 1$.

Proof. We give a proof for $W^-(1, x; t_1, z)$. The case of W^+ can be proved similarly. Note that

$$W^-(0, x; t_1, z) \geq U(t_1 - 2\tau, x; z) \quad \forall x, z \in \mathbb{R}^N.$$

It then follows that

$$W^-(1, x; t_1, z) > U(1 + t_1 - 2\tau, x; z) \quad \forall x, z \in \mathbb{R}^N.$$

Take an $\epsilon_1 \in (0, \epsilon_0]$. By Lemma 5.2, for any $\epsilon \in (0, \epsilon_1]$,

$$W^-(1, x; t_1, z) > (1 - \epsilon)U(1 + t_1 - 2\tau + 3l\epsilon, x; z) \quad \forall x \cdot \xi - c(t_1 + 1) \leq -M(\epsilon_1), \quad z \in \mathbb{R}^N.$$

We claim that for $0 < \epsilon \ll 1$,

$$W^-(1, x; t_1, z) > (1 - \epsilon)U(1 + t_1 - 2\tau + 3l\epsilon, x; z) \quad \forall x \cdot \xi - c(t_1 + 1) \in [-M(\epsilon_1), M], \quad z \in \mathbb{R}^N.$$

In fact, let $W(t, x; z) = W^-(t, x; t_1, z) - U^+(t + t_1 - 2\tau, x; z)$ and

$$\begin{aligned} h = & \inf_{t \in [0, 1], x, z \in \mathbb{R}^N} \{ [W^-(t, x; t_1, z)f(x + z, u(t, x; u_{0, z}, z)) \\ & - U(t + t_1 - 2\tau, x; z)f(x + z, U(t + t_1 - 2\tau, x; z))] \\ & \cdot \frac{1}{W^-(t, x; t_1, z) - U(t + t_1 - 2\tau, x; z)} \}. \end{aligned}$$

Then

$$W(0, x; z) = \begin{cases} U(t_1 - \tau, x; z) - U(t_1 - 2\tau, x; z) & \text{for } x \cdot \xi - ct_1 > M \\ 0 & \text{for } x \cdot \xi - ct_1 \leq M \end{cases}$$

and

$$W_t(t, x; z) \geq \int_{\mathbb{R}^N} k(y - x)W(t, y; z)dy - W(t, x; z) + hW(t, x; z) \quad \forall t \in [0, 1], \quad x, z \in \mathbb{R}^N.$$

It then follows that

$$W(1, \cdot; z) \geq e^{-1+h}(W(0, \cdot; z) + \mathcal{K}W(0, \cdot; z) + \frac{\mathcal{K}^2}{2!}W(0, \cdot; z) + \cdots),$$

where $\mathcal{K}W(0, \cdot; z)$ is defined as in (1.5) with u being replaced by $W(0, \cdot; z)$. By Lemma 4.2, there are $\tilde{\sigma} > 0$ and $\tilde{M} > 0$ such that

$$U(t_1 - \tau, x; z) - U(t_1 - 2\tau, x; z) \geq \tilde{\sigma} \quad \forall x, z \in \mathbb{R}^N \quad \text{with } \tilde{M} \leq x \cdot \xi - ct_1 \leq \tilde{M} + 1. \quad (5.5)$$

This implies that

$$W(1, x; z) \geq U(1 + t_1 - 2\tau + 3l\epsilon, x; z) - U(1 + t_1 - 2\tau, x; z) \quad (5.6)$$

for $x \cdot \xi - c(t_1 + 1) \in [-M(\epsilon_1), M]$ and $z \in \mathbb{R}^N$ provided that $0 < \epsilon \ll 1$. By (5.5) and (5.6), we have

$$\begin{aligned} W^-(1, x; t_1, z) &= W(1, x; z) + U(1 + t_1 - 2\tau, x; z) \\ &\geq U(1 + t_1 - 2\tau + 3l\epsilon, x; z) \\ &\geq (1 - \epsilon)U(1 + t_1 - 2\tau + 3l\epsilon, x; z) \end{aligned}$$

for $x \cdot \xi - c(1 + t_1) \leq M$ and $z \in \mathbb{R}^N$ provided that $0 < \epsilon \ll 1$. □

Proof of Theorem 2.2. (1) Let

$$A^+ = \{\tau \geq 0 \mid \limsup_{t \rightarrow \infty} \sup_{x, z \in \mathbb{R}^N} \frac{U_1(t, x; z)}{U(t + 2\tau, x; z)} \leq 1\}$$

and

$$A^- = \{\tau \geq 0 \mid \liminf_{t \rightarrow \infty} \inf_{x, z \in \mathbb{R}^N} \frac{U_1(t, x; z)}{U(t - 2\tau, x; z)} \geq 1\}.$$

By Lemma 5.4, $A^\pm \neq \emptyset$. Let

$$\tau^+ = \inf\{\tau \mid \tau \in A^+\}, \quad \tau^- = \inf\{\tau \mid \tau \in A^-\}.$$

We first claim that $\tau^\pm \in A^\pm$. In fact, let $\tau_n \in A^+$ be such that $\tau_n \rightarrow \tau^+$. Then for any $0 < \epsilon < 1$, there are $t_n \rightarrow \infty$ such that

$$\frac{U_1(t, x; z)}{U(t + 2\tau_n, x; z)} \leq 1 + \epsilon \quad \forall x, z \in \mathbb{R}^N, \quad t \geq t_n$$

and

$$\frac{U(t + 2\tau^+, x; z) - U(t + 2\tau_n, x; z)}{U(t + 2\tau_n, x; z)} > -\epsilon \quad \forall n \gg 1, \quad t \in \mathbb{R}, \quad x, z \in \mathbb{R}^N.$$

Observe that

$$\frac{U_1(t, x; z)}{U(t + 2\tau^+, x; z)} = \frac{U_1(t, x; z)}{U(t + 2\tau_n, x; z)} \frac{U(t + 2\tau_n, x; z)}{U(t + 2\tau^+, x; z)}$$

and

$$\begin{aligned} \frac{U(t + 2\tau_n, x; z)}{U(t + 2\tau^+, x; z)} &= \frac{1}{1 + \frac{U(t + 2\tau^+, x; z) - U(t + 2\tau_n, x; z)}{U(t + 2\tau_n, x; z)}} \\ &\leq \frac{1}{1 - \epsilon} \\ &\leq 1 + \epsilon \quad \forall n \gg 1. \end{aligned}$$

Fix $n \gg 1$. Then

$$\sup_{x, z \in \mathbb{R}^N} \frac{U_1(t, x; z)}{U(t - 2\tau^-, x; z)} \leq (1 + \epsilon)^2 \quad \forall t \geq t_n.$$

This implies that $\tau^+ \in A^+$. Similarly, we have $\tau^- \in A^-$.

Next we claim that $\tau^\pm = 0$. Assume that $\tau^- > 0$. Note that

$$\liminf_{t \rightarrow \infty} \inf_{x, z \in \mathbb{R}^N} \frac{U_1(t, x; z)}{U(t - 2\tau^-, x; z)} \geq 1.$$

Hence for any $\bar{\epsilon} > 0$, there is $t_0 > 0$ such that

$$\frac{U_1(t_0, x; z)}{U(t_0 - 2\tau^-, x; z)} \geq 1 - \bar{\epsilon} \quad \forall x, z \in \mathbb{R}^N.$$

This implies that

$$U_1(t_0, x; z) \geq (1 - \bar{\epsilon})U(t_0 - 2\tau^-, x; z) \geq U^+(t_0 - 2\tau^-, x; z) - \hat{\epsilon}$$

where $\hat{\epsilon} = \bar{\epsilon} \max_{t, x, z} U^+(t, x, z)$. By Lemma 5.3, for $x \cdot \xi - ct_0 \geq M := C(\tau^-/2)$,

$$U_1(t_0, x; z) \geq U(t_0 - \tau^-, x; z).$$

This implies that

$$U_1(t_0, x; z) \geq U(t_0 - 2\tau^-, x; z)(1 - \zeta(x \cdot \xi - ct_0 - M)) + U(t_0 - \tau^-, x; z)\zeta(x \cdot \xi - ct_0 - M) - \hat{\epsilon}.$$

Note that there is $K > 0$ such that $U_1(t, x; z) + \hat{\epsilon}e^{Kt}$ is a super-solution of (2.5) for $t \in [0, 1]$ provided that $0 < \hat{\epsilon} \ll 1$. By Lemma 5.5, for $0 < \bar{\epsilon} \ll 1$ and $0 < \epsilon \ll 1$,

$$U_1(t_0 + 1, x; z) + \hat{\epsilon}e^K \geq (1 - \epsilon)U(t_0 + 1 - 2\tau^- + 3l\epsilon, x; z) \quad \forall x \cdot \xi - c(t_0 + 1) \leq M, z \in \mathbb{R}^N,$$

where l is as in Lemma 5.1. Then for $0 < \bar{\epsilon} \ll \epsilon \ll 1$,

$$U_1(t_0 + 1, x; z) \geq (1 - 2\epsilon)U(t_0 + 1 - 2\tau^- + 3l\epsilon, x; z) \quad \forall x \cdot \xi - c(t_0 + 1) \leq M, z \in \mathbb{R}^N.$$

By Lemma 5.3 again, for $x \cdot \xi - c(t_0 + 1) \geq M$, $z \in \mathbb{R}^N$, and $0 < \epsilon \ll 1$,

$$\begin{aligned} U_1(t_0 + 1, x; z) &> U(t_0 + 1 - \tau^-, x; z) \\ &\geq (1 - 2\epsilon)U(t_0 + 1 - \tau^-, x; z) \\ &\geq (1 - 2\epsilon)U(t_0 + 1 - 2\tau^- + 3l\epsilon, x; z). \end{aligned}$$

Therefore for $0 < \epsilon \ll 1$,

$$U_1(t_0 + 1, x; z) \geq (1 - 2\epsilon)U(t_0 + 1 - 2\tau^- + 3l\epsilon, x; z) \quad \forall x, z \in \mathbb{R}^N.$$

By Lemma 5.1,

$$U_1(t_0 + t + 1, x; z) \geq (1 - 2\epsilon e^{-\tau t})U(t_0 + 1 + t - 2\tau^- + 2l\epsilon e^{-\eta t} + l\epsilon, x; z) \quad \forall t \geq 0, x, z \in \mathbb{R}^N.$$

It then follows that

$$\tau^- - \frac{l\epsilon}{2} \in A^-.$$

this is a contradiction. Therefore $\tau^- = 0$. Similarly, we have $\tau^+ = 0$.

We now prove that $\Phi_1(x, z) = \Phi(x, z)$. Recall that $U_1(t, x; z) = \Phi_1(x - ct\xi, z + ct\xi)$ and $U(t, x; z) = \Phi(x - ct\xi, z + ct\xi)$. Hence

$$\begin{aligned} \inf_{x, z \in \mathbb{R}^N} \frac{U_1(t, x; z)}{U(t, x; z)} &= \inf_{x, z \in \mathbb{R}^N} \frac{\Phi_1(x - ct\xi, z + ct\xi)}{\Phi(x - ct\xi, z + ct\xi)} \\ &= \inf_{x, z \in \mathbb{R}^N} \frac{\Phi_1(x, z)}{\Phi(x, z)} \end{aligned}$$

and

$$\begin{aligned} \sup_{x, z \in \mathbb{R}^N} \frac{U_1(t, x; z)}{U(t, x; z)} &= \sup_{x, z \in \mathbb{R}^N} \frac{\Phi_1(x - ct\xi, z + ct\xi)}{\Phi(x - ct\xi, z + ct\xi)} \\ &= \sup_{x, z \in \mathbb{R}^N} \frac{\Phi_1(x, z)}{\Phi(x, z)} \end{aligned}$$

This together with $\tau^\pm = 0$ implies that

$$\inf_{x, z \in \mathbb{R}^N} \frac{\Phi_1(x, z)}{\Phi(x, z)} = \sup_{x, z \in \mathbb{R}^N} \frac{\Phi_1(x, z)}{\Phi(x, z)} = 1.$$

We then must have $\Phi_1(x, z) \equiv \Phi(x, z)$.

(2) Let $\Phi_1(x, z) = \Phi^-(x, z) (= U^-(0, x; z))$. By (1), $\Phi^-(x, z) = \Phi(x, z)$. Recall that $\Phi^-(x, z)$ is lower semi-continuous and $\Phi^+(x, z)$ is upper semi-continuous. We then must have that $\Phi(x, z)$ is continuous in $(x, z) \in \mathbb{R}^N \times \mathbb{R}^N$. □

Corollary 5.1. *Let $\Phi(x, z)$ be as above. Then*

$$\lim_{\tau \rightarrow \infty} u(\tau, x; \underline{u}(0, \cdot; z, \tau, d_1, b), z) = \lim_{\tau \rightarrow \infty} u(\tau, x; \bar{u}(0, \cdot; z, \tau, d_2), z) = \Phi(x, z)$$

for all $d_1 \gg 1$, $d_2 > 0$, $0 < b \ll 1$, and $x, z \in \mathbb{R}^N$.

Proof. By the arguments of Theorem 2.1(3), for any $d_1 \gg 1$ and $0 < b \ll 1$,

$$\lim_{\tau \rightarrow \infty} u(\tau, x; \underline{u}(0, \cdot; z, \tau, d_1, b), z) = \Phi^+(x, z) (= \Phi(x, z)) \quad \forall x, z \in \mathbb{R}^N,$$

and for any $d_2 \gg 1$,

$$\lim_{\tau \rightarrow \infty} u(\tau, x; \bar{u}(0, \cdot; z, \tau, d_2), z) = \Phi^-(x, z) (= \Phi(x, z)) \quad \forall x, z \in \mathbb{R}^N.$$

The corollary then follows. □

6 Stability of Traveling Wave Solutions and Proof of Theorem 2.3

In this section, we investigate the stability of traveling wave solutions of (1.2) and prove Theorem 2.3.

Throughout this section, we fix $\xi \in S^{N-1}$ and $c > c^*(\xi)$. Let μ^* be such that

$$c^*(\xi) = \frac{\lambda_0(\mu^*, \xi, a_0)}{\mu^*} < \frac{\lambda_0(\tilde{\mu}, \xi, a_0)}{\tilde{\mu}} \quad \forall \tilde{\mu} \in (0, \mu^*).$$

We fix $c > c^*(\xi)$ and $\mu \in (0, \mu^*)$ with $\frac{\lambda_0(\mu, \xi, a_0)}{\mu} = c$. Let $U(t, x; z) = U^+(t, x; z)$, where $U^+(t, x; z)$ is as in section 4. We put $u(t, x) = u(t, x; u_0, 0)$, where u_0 is as in Theorem 2.3, and put $U(t, x) = U^+(t, x; 0)$. We can prove Theorem 2.3 by Lemmas 6.1-6.3 and the similar arguments in Theorem 2.2. Here we only state these lemmas without proofs, which can be proved by properly modifying the arguments in their counterparts of Lemmas 5.3-5.5.

Lemma 6.1. *For any $\epsilon > 0$, there exists a constant $C_0(\epsilon) \geq 1$ such that*

$$u(t - 2\epsilon, x) \leq U(t, x) \leq u(t + 2\epsilon, x) \quad \forall x \cdot \xi - ct \geq C_0(\epsilon), \quad t \geq 2\epsilon.$$

Lemma 6.2. *Let ϵ_0 , η , and l be as in Lemma 5.1. For given $\epsilon \in (0, \epsilon_0)$, there are $t_{\pm} > 0$ and $\tau_{\pm} > 0$ such that*

$$(1 - \epsilon e^{-\eta(t-t_-)})U(t - \tau_- + l\epsilon e^{-\eta(t-t_-)}, x) \leq u(t, x) \leq (1 + \epsilon e^{-\eta(t-t_+)})U(t + \tau_+ - l\epsilon e^{-\eta(t-t_+)}, x)$$

for all $x \in \mathbb{R}^N$ and $t \geq \max\{t_-, t_+\}$.

Lemma 6.3. *Let $\tau > 0$, $t_1 > 0$, and $M \in \mathbb{R}$ be given. Suppose that $w^{\pm}(\cdot, x; t_1)$ are the solution of (1.2) for $t \geq 0$ with the initial conditions*

$$w^{\pm}(0, x; t_1) = U(t_1 \pm \tau, x)\varsigma(x - ct_1 - M) + U(t_1 \pm 2\tau, x)(1 - \varsigma(x - ct_1 - M)) \quad \forall x \in \mathbb{R}^N,$$

where $\varsigma(s) = 0$ for $s \leq 0$ and $\varsigma(s) = 1$ for $s > 0$. Then

$$\begin{aligned} w^+(1, x; t_1) &\leq (1 + \epsilon)U(t_1 + 1 + 2\tau - 3l\epsilon) \\ w^-(1, x; t_1) &\geq (1 - \epsilon)U(t_1 + 1 - 2\tau + 3l\epsilon), \end{aligned}$$

for all $x \cdot \xi - ct_1 \leq M + c$ and $0 < \epsilon \ll 1$.

References

- [1] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, *Adv. Math.*, **30** (1978), pp. 33-76.

- [2] P. Bates and G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, *J. Math. Anal. Appl.* **332** (2007), pp. 428-440.
- [3] H. Berestycki, F. Hamel, and N. Nadirashvili, The speed of propagation for KPP type problems, I - Periodic framework, *J. Eur. Math. Soc.* **7** (2005), pp. 172-213.
- [4] H. Berestycki, F. Hamel, and N. Nadirashvili, The speed of propagation for KPP type problems, II - General domains, *J. Amer. Math. Soc.* **23** (2010), no. 1, pp. 1-34
- [5] H. Berestycki, F. Hamel, and L. Roques, Analysis of periodically fragmented environment model: II - Biological invasions and pulsating traveling fronts, *J. Math. Pures Appl.* **84** (2005), pp. 1101-1146.
- [6] E. Chasseigne, M. Chaves, and J. D. Rossi, Asymptotic behavior for nonlocal diffusion equations, *J. Math. Pures Appl.*, **86** (2006) 271-291.
- [7] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, *Adv. Differential Equations*, **2** (1997), pp. 125-160.
- [8] X. Chen and J.-S. Guo, Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations, *J. Diff. Eq.*, **184** (2002), no. 2, pp. 549-569.
- [9] X. Chen and J.-S. Guo, Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics, *Math. Ann.*, **326** (2003), no. 1, pp. 123-146.
- [10] C. Cortazar, M. Elgueta, and J. D. Rossi, Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions, *Israel J. of Math.*, **170** (2009), 53-60.
- [11] J. Coville, On uniqueness and monotonicity of solutions of non-local reaction diffusion equation, *Annali di Matematica* **185**(3) (2006), pp. 461-485
- [12] J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators, *J. Differential Equations* **249** (2010), 2921-2953.
- [13] J. Coville and L. Dupaigne, Propagation speed of travelling fronts in non local reaction-diffusion equations, *Nonlinear Analysis* **60** (2005), pp. 797 - 819
- [14] J. Coville, J. Dávila, and S. Martínez, Existence and uniqueness of solutions to a nonlocal equation with monostable nonlinearity, *SIAM J. Math. Anal.* **39** (2008), pp. 1693-1709.
- [15] J. Coville, J. Dávila, S. Martnez, Nonlocal anisotropic dispersal with monostable nonlinearity, *J. Differential Equations* **244** (2008), pp. 3080 - 3118.
- [16] P. C. Fife and J. B. Mcleod, The approach of solutions of nonlinear diffusion equations to traveling front solutions, *Arch. Ration. Mech. Anal.*, **65** (1977), pp. 335-361.

- [17] R. Fisher, The wave of advance of advantageous genes, *Ann. of Eugenics*, **7**(1937), pp. 335-369.
- [18] M. Freidlin and J. Gärtner, On the propagation of concentration waves in periodic and random media, *Soviet Math. Dokl.*, **20** (1979), pp. 1282-1286.
- [19] J. García-Melán and J. D. Rossi, On the principal eigenvalue of some nonlocal diffusion problems, *J. Differential Equations*, **246** (2009) 2138.
- [20] M. Grinfeld, G. Hines, V. Hutson, K. Mischaikow, and G. T. Vickers, Non-local dispersal, *Differential Integral Equations*, **18** (2005), pp. 1299-1320.
- [21] J.-S. Guo and F. Hamel, Front propagation for discrete periodic monostable equations, *Math. Ann.*, **335** (2006), no. 3, pp. 489–525.
- [22] J.-S. Guo and C.-C. Wu, Uniqueness and stability of traveling waves for periodic monostable lattice dynamical system, *J. Differential Equations* **246** (2009), pp. 3818-3833.
- [23] F. Hamel, Qualitative properties of monostable pulsating fronts : exponential decay and monotonicity, *J. Math. Pures Appl.* **89** (2008), pp. 355-399.
- [24] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math. **840**, Springer-Verlag, Berlin, 1981.
- [25] G. Hetzer, T. Nguyen, and W. Shen, Coexistence and Extinction in the Volterra-Lotka Competition Model with Nonlocal Dispersal, preprint.
- [26] G. Hetzer, W. Shen, and A. Zhang, Effects of Spatial Variations and Dispersal Strategies on Principal Eigenvalues of Dispersal Operators and Spreading Speeds of Monostable Equations, *Rocky Mountain Journal of Mathematics*, to appear.
- [27] J. Huang and W. Shen, Speeds of Spread and propagation for KPP Models in Time Almost and Space Periodic Media, *SIAM J. Appl. Dynam. Syst.*, **8** (2009), pp. 790-821.
- [28] W. Hudson and B. Zinner, Existence of traveling waves for reaction diffusion equations of Fisher type in periodic media, Boundary value problems for functional-differential equations, 187–199, World Sci. Publ., River Edge, NJ, 1995.
- [29] W. Hudson and B. Zinner, Existence of traveling waves for a generalized discrete Fisher's equation, *Comm. Appl. Nonlinear Anal.*, **1** (1994), no. 3, pp. 23-46.
- [30] V. Hutson and M. Grinfeld, Non-local dispersal and bistability, *Euro. J. Appl. Math* **17** (2006), pp. 221-232.
- [31] V. Hutson, S. Martinez, K. Mischaikow, and G.T. Vickers, The evolution of dispersal, *J. Math. Biol.* **47** (2003), pp. 483-517.

- [32] V. Hutson, W. Shen and G.T. Vickers, Spectral theory for nonlocal dispersal with periodic or almost-periodic time dependence, *Rocky Mountain Journal of Mathematics* **38** (2008), pp. 1147-1175.
- [33] Y. Kametaka, On the nonlinear diffusion equation of Kolmogorov-Petrovskii- Piskunov type, *Osaka J. Math.*, **13** (1976), pp. 11-66.
- [34] C.-Y. Kao, Y. Lou, and W. Shen, Random dispersal vs non-Local dispersal, *Discrete and Continuous Dynamical Systems*, **26** (2010), no. 2, pp. 551-596
- [35] A. Kolmogorov, I. Petrowsky, and N. Piskunov, A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem. *Bjul. Moskovskogo Gos. Univ.*, **1** (1937), pp. 1-26.
- [36] W.-T. Li, Y.-J. Sun, Z.-C. Wang, Entire solutions in the Fisher-KPP equation with nonlocal dispersal, *Nonlinear Analysis, Real World Appl.*, **11** (2010), no. 4, pp. 2302-2313.
- [37] X. Liang and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.*, **60** (2007), no. 1, pp. 1-40.
- [38] X. Liang and X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, *Journal of Functional Analysis*, to appear.
- [39] X. Liang, Y. Yi, and X.-Q. Zhao, Spreading speeds and traveling waves for periodic evolution systems, *J. Diff. Eq.*, **231** (2006), no. 1, pp. 57-77.
- [40] R. Lui, Biological growth and spread modeled by systems of recursions, *Math. Biosciences*, **93** (1989), pp. 269-312.
- [41] G. Lv and M. Wang, Existence and stability of traveling wave fronts for nonlocal delayed reaction diffusion systems, preprint.
- [42] G. Nadin, Traveling fronts in space-time periodic media, *J. Math. Pures Appl.*, **(9) 92** (2009), pp. 232-262.
- [43] J. Nolen, M. Rudd, and J. Xin, Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds, *Dynamics of PDE*, **2** (2005), pp. 1-24.
- [44] J. Nolen and J. Xin, Existence of KPP type fronts in space-time periodic shear flows and a study of minimal speeds based on variational principle, *Discrete and Continuous Dynamical Systems*, **13** (2005), pp. 1217-1234.
- [45] S. Pan, W.-T. Li, and G. Lin, Existence and stability of traveling wavefronts in a nonlocal diffusion equation with delay, *Nonlinear Analysis: Theory, Methods & Applications*, **72** (2010), 3150-3158.

- [46] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag New York Berlin Heidelberg Tokyo, 1983.
- [47] L. Ryzhik and A. Zlatoš, KPP pulsating front speed-up by flows, *Commun. Math. Sci.*, **5** (2007), no. 3, pp. 575-593.
- [48] D. H. Sattinger, On the stability of waves of nonlinear parabolic systems, *Advances in Math.*, **22** (1976), pp. 312-355.
- [49] W. Shen, Variational principle for spatial spreading speeds and generalized propagating speeds in time almost and space periodic KPP models, *Trans. Amer. Math. Soc.*, to appear.
- [50] W. Shen, Spreading and generalized propagating speeds of discrete KPP models in time varying environments, *Frontiers of Mathematics in China*, **4(3)** (2009), pp. 523-562.
- [51] W. Shen, Existence, uniqueness, and stability of generalized traveling waves in time dependent monostable equations, to appear in *J. Dynam. Diff. Equations*.
- [52] W. Shen, Existence of Generalized Traveling Waves in Time Recurrent and Space Periodic Monostable Equations, to appear in *Journal of Applied Analysis and Computation*.
- [53] W. Shen and G. T. Vickers, Spectral theory for general nonautonomous/random dispersal evolution operators, *J. Differential Equations*, **235** (2007), pp. 262-297.
- [54] W. Shen and A. Zhang, Spreading Speeds for Monostable Equations with Nonlocal Dispersal in Space Periodic Habitats, *Journal of Differential Equations* **249** (2010), 747-795.
- [55] W. Shen and A. Zhang, Stationary Solutions and Spreading Speeds of Nonlocal Monostable Equations in Space Periodic Habitats, submitted.
- [56] K. Uchiyama, The behavior of solutions of some nonlinear diffusion equations for large time, *J. Math. Kyoto Univ.*, **183** (1978), pp. 453-508.
- [57] H. F. Weinberger, Long-time behavior of a class of biology models, *SIAM J. Math. Anal.*, **13** (1982), pp. 353-396.
- [58] H. F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, *J. Math. Biol.*, **45** (2002), pp. 511-548.
- [59] J. Wu and X. Zou, Asymptotic and periodic boundary value problems of mixed FDEs and wave solutions of lattice differential equations, *J. Diff. Eq.*, **135** (1997), no. 2, pp. 315-357.
- [60] B. Zinner, G. Harris, and W. Hudson, Traveling wavefronts for the discrete Fisher's equation, *J. Diff. Eq.*, **105** (1993), no. 1, pp. 46-62.